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Semiclassical stability analysis of a two-photon laser including spatial variation of the cavity field

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Abstract. We investigate the dynamics of a two-photon laser under conditions where the spatial variation of the cavity field along the cavity axis is important. The model assumes pumping to the upper state of the two-photon transition. We consider the Maxwell-Bloch equations on the basis of which we study the stability analysis of the steady state of the system. The system is taken to be contained in a ring-laser cavity. Asymptotic expansion of the eigenvalue and analytic information are obtained in some realistic limits, such as very large reflectivity, very small cavity losses, or very small population relaxation rate. The results are illustrated with an application to a specific atomic system (potassium) as an amplifying medium.

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1 Introduction

One of the main obstacles in the realization of the standard form of the two-photon laser is a technological one; namely the difficulty in constructing a cavity in which the modes are sufficiently separated in frequency for the two-photon gain to prevail over the single-photon gain in an adjacent frequency corresponding to a dipole transition from the upper state to an intermediate one of opposite parity. If a cavity mode happens to be sufficiently near resonance with one-photon transition between the upper (pumped) and the intermediate level, the gain into that transition will take over. The basic simple model of interaction between matter and radiation comprises two-level atoms interacting with a single-mode electromagnetic field in a lossless cavity. In spite of its simplicity, the model is intrinsically nonlinear with the atom-field coupling being the coefficient of this nonlinearity.

A brief history of the topic may be useful here. Two-photon lasers are a recurrent theme in the literature and have attracted considerable theoretical interest semiclassically by Wang and Haken [1] as well as quantum mechanically [2–4]. On the experimental side, a two-photon laser has been realized and studied extensively in the microwave region of the spectrum [5] and in the optical regime by Mossberg and collaborators [6,7] in a cleaver, slightly different scheme, where the atomic pump transition is between levels of dressed atoms. Recent work [8] in view of continuing technological improvements in micro-cavities even at optical frequencies has motivated the examination of certain aspects of the two-photon laser theory that are fundamental to the process. These aspects have their counterpart in the usual single-photon laser, but rather different behavior is to be expected in the two-photon case, owing to the essential nonlinearity of the process even at weak signals. We have here in mind a degenerate two-photon laser with the atom pumped to the upper state connected to the lower one of the lasing transition by a two-photon process. Although not realized as yet in this pure form, it probably is a matter of short time before that occurs [2–4]. The situation here is somewhat different from the dressed states scheme that has already been demonstrated experimentally some time ago [6,7]. Most laser-related systems derive their feedback from a resonator structure. The emergence of unstable behavior in this case can be ascribed to the development of amplitude oscillations in an excited but previously stable mode or to the growth of new cavity resonances. For this reason, all previous investigations of this problem have focused separately on what can be conveniently identified as single-mode and multimode instabilities. Instabilities of the first type involve only the cavity mode that lies nearest to, but is not necessarily in resonance with, the atomic transition, or the carrier frequency of an incident field, for externally driven system. These phenomena can be analyzed in the context of a single-mode model. The

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second type involves the evolution of out-of-resonance cavity modes and therefore requires a multimode treatment for a correct description.

Most linear-stability studies avoid dealing with the spatial dependence ($\partial E/\partial z$ term in the field equation) [9] and employ what has become known as the mean-field limit. By taking this limit, the field equation is spilt into a set of purely temporal equations, each of which governs the evolution of a single longitudinal mode. Although limited in its scope, the mean-field approximation has proved to be quite successful in studying the stability of lasers. However, one should be aware that its predictions for any real laser need to be checked against numerical solutions of the exact Maxwell-Bloch equations.

The issue we have in mind has to do with the steady-state behavior and its stability analysis of the system, taking into account the spatial variation of the cavity field along the cavity axis. This is most conveniently accomplished in a semiclassical formalism in terms of the Maxwell-Bloch equations taking into account the spatial dependence. Related treatments based on either simple rate equations [10], discussing threshold conditions, or the Maxwell-Bloch equations without the spatial dependence, have been presented in the literature [11–13]. What we study and present below is essentially the generalization of the complete Maxwell-Bloch equations, usually employed in the single-photon laser theory, to the two-photon case. We have found it most convenient to use a formulation presented some time ago by Narducci in the semiclassical theory of the single-photon laser [14]. We present in this paper an analysis which has inspired by the comparison between the linear stability analysis technique and the so-called weak sideband approach, first introduced by Casperson [15] and further elaborated by Hendow and Sargent [16] and by Boyd, Hillman, and Stroud [17].

2 Derivation of equations

Maxwell-Bloch equations for a degenerate two-photon laser with the atom pumped to the upper state connected to the lower one of the lasing transition by a two-photon process and its steady state have been derived and discussed previously in [18]. We present a brief sketch of the derivation of the equations in order to specify the assumptions used in the derivation and in order to have at hand for discussion the terms relevant to the stability analysis of the steady state.

We consider the coupled set of Maxwell-Bloch equations, in the usual rotating wave approximation, which govern our two-level atom when the dipole forbidden transition is replaced by a two-photon one. We consider the degenerate case, in which pairs of photons with the same frequency are created or absorbed, and we analyze the stability of the steady state. We assume a collection of identical homogeneously broadened two-level atoms, with energies E_1 and E_2 such that $(E_2 > E_1)$ with $E_2 - E_1 = \hbar\omega_\mathrm{a}$, ω_a the atomic transition frequency and a generated unidirectional single-mode classical electric field

$$E(z,t) = \frac{1}{2}\left\{E_0 \mathrm{e}^{\mathrm{i}(k_\mathrm{c} z - \omega_\mathrm{c} t)} + \mathrm{c.c.}\right\}, \tag{1}$$

inside a ring cavity. Here E_0 is the real field amplitude, k_c the wave-number, z the cavity axial direction and ω_c represents the unloaded cavity frequency. The atoms interact with the field in the dipole approximation *via* a two-photon transition, where these states are assumed to have the same parity, and thus are not connected by a one-photon transition.

Adopting the plane-wave approximation, we reduce the Maxwell-Bloch equations to [18]

$$\frac{\partial F}{\partial z} + \frac{1}{c}\frac{\partial F}{\partial t} = -\alpha P F^*, \tag{2}$$

$$\frac{\partial P}{\partial t} = -(\gamma_1 + \mathrm{i}\delta_\mathrm{ac})\bar{P} - \gamma_1 F^2 D, \tag{3}$$

$$\frac{\partial D}{\partial t} = \gamma_2 \left\{\frac{1}{2}(PF^{*2} + \bar{P}^* F^2) - D + 1\right\}, \tag{4}$$

where F, P and D are the normalized output field, two-photon polarization and population difference, respectively, $(F = \sqrt{\mu^{(2)}/\hbar\gamma_1\gamma_2}\bar{E}_0)$, $\mu^{(2)}$ the effective dipole matrix element for the two-photon transition, γ_1 and γ_2 are the decay rates of two-photon polarization and population difference, respectively. α denotes the unsaturated gain constant per unit length of the active medium $(\alpha = 2N\omega_\mathrm{c}(\mu^{(2)})^{2/3}/2c\hbar\varepsilon_0\gamma_1)$, where N is the number of atoms per unit volume, ε_0 the vacuum electric permeability and c the speed of light. We denote by $\delta_\mathrm{ac} = \omega_\mathrm{a} - 2\omega_\mathrm{c}$ the detuning of the cavity mode from two-photon resonance. Maxwell-Bloch equations (2–4) have been derived by assuming an effective Hamiltonian, *i.e.*, by assuming a pure two-photon interaction between the two-level atom and electromagnetic field. This approach neglects residual effects of any largely detuned one-photon transitions between the lasing levels and other atomic levels [19]. A more precise approach consists in assuming an exact or microscopic interaction Hamiltonian that describes the interaction of the electromagnetic field with a three-level cascade atomic scheme [20,21]. When the intermediate atomic level is far from one-photon resonance the one-photon coherence can be adiabatically eliminated and the resulting two-photon laser equations are similar to the present equations but include three additional detuning terms describing frequency shifts.

The model is completed by appropriate boundary conditions which, in the case of a traveling wave ring-cavity resonator, take the form

$$F(0,t) = RF(L, t - (\Lambda - L)/c), \tag{5}$$

where L is the length of the active medium; while the full length of the ring resonator is Λ. Physically R measures the loss of the field amplitude from the exit face of the amplifying medium to its entrance face.

3 Steady state

In order to gain some physical understanding of the process and discuss some aspects of the threshold conditions, we analyze first the steady-state behavior of the system. To study the steady state, we consider the equations in the long-time limit by setting the time derivatives equal to zero, for an active medium detuned by an arbitrary amount $\delta_{\rm ac}$ from the resonant cavity mode. Under these conditions, the output field is expected to oscillate with a carrier frequency $\omega_{\rm L}$ which is neither equal to $\omega_{\rm c}$ nor $\omega_{\rm a}/2$, but to some intermediate value determined by the cavity and atomic parameters. For this reason, we look for steady-state solutions of the type

$$F(z,t) = F_{\rm st}(z)\mathrm{e}^{-\mathrm{i}\delta\omega t}, \qquad (6)$$

$$P(z,t) = P_{\rm st}(z)\mathrm{e}^{-\mathrm{i}2\delta\omega t}, \qquad (7)$$

$$D(z,t) = D_{\rm st}(z), \qquad (8)$$

where $\delta\omega$ is the frequency offset of the operating laser line from the resonant mode (*i.e.* $\delta\omega = \omega_{\rm L} - \omega_{\rm c}$). The atomic variables can be determined at once as functions of the stationary field profile

$$P_{\rm st}(z) = -F_{\rm st}^2(z)\frac{1-\mathrm{i}\Delta}{1+\Delta^2+|F_{\rm st}(z)|^4}, \qquad (9)$$

$$D_{\rm st}(z) = \frac{1+\Delta^2}{1+\Delta^2+|F_{\rm st}(z)|^4}, \qquad (10)$$

where the detuning parameter Δ is defined as $\Delta = (\delta_{\rm ac} - 2\delta\omega)/\gamma_1$. The steady state polarization and the field envelope are generally out of phase from one another by an amount that depends on the detuning $\delta_{\rm ac}$ and the position of the operating laser line. The steady state population difference (inversion) saturates at high intensity levels in the sense that $D_{\rm st} \to 0$ as $|F_{\rm st}| \to \infty$. To determine the value of the output field and the form of its longitudinal profile in steady state, it is convenient to represent the field amplitude in terms of its modulus only,

$$\frac{\mathrm{d}|F_{\rm st}(z)|}{\mathrm{d}z} = \frac{\alpha|F_{\rm st}(z)|^3}{1+\Delta^2+|F_{\rm st}(z)|^4}\,. \qquad (11)$$

To make the analysis in the present paper as simple as possible we shall not consider the phase equation [18]. However, the effects of the phase variation should be taken into consideration for a more elaborate discussion. The boundary condition, expressed in terms of the field modulus is given by $F_{\rm st}(0) = RF_{\rm st}(L)$. The output laser intensity can be calculated as [18],

$$|F_{\rm st}(L)|^2 = \frac{2\alpha L}{1-R^2}|F_{\rm st}(L)|^2 - \frac{1+\Delta_j^2}{R^2}, \qquad (12)$$

where $\Delta_j = (\delta_{\rm ac} - 2\delta\omega_j)/\gamma_1$, $\delta\omega_j$ is the operating laser frequency. Equation (12) has two roots and at laser threshold the intensity is not vanishing. There is coexistence of three solutions (above threshold): the trivial and two other solutions with intensity different from zero. One solution grows with the pumping parameter up to an asymptotic value for pumping going to infinity. The other solution decreases towards the zero solution as the pumping grows to infinity. This means that the threshold is not a second order phase transition as in the case of single photon lasers.

The quantity $c|\ln R|/\gamma_1\Lambda$ represents the decay rate of the cavity field, and $2\pi c/\Lambda$ is the spacing between adjacent cavity resonances. After introducing the abbreviations $K = c|\ln R|/\Lambda$, $\alpha_1 = 2\pi c/\Lambda$, we obtain

$$\delta\omega_j = \omega_{\rm L} - \omega_{\rm c} = \frac{K\delta_{\rm ac} + \alpha_1\gamma_1 j}{\gamma_1 + 2K}, \qquad (13)$$

where the sub-index j reminds us of the possible existence of multiple solutions. This is the well known mode-pulling formula. It shows that the laser operating frequency is a weighted average of the atomic resonant frequency and the frequency of one of the cavity modes.

4 Linear stability analysis

The general stability analysis of the Maxwell-Bloch equations (2–4) is a rather difficult problem. The main source of complication originates from the spatial dependence of the field and of the atomic variables. In an attempt to get around this problem, most linear stability analysis have been carried out within the uniform field limit. While this may not appear to be a very realistic approach, there are good reasons, in fact, why useful information can be extracted even from this limiting case: (i) we can reformulate the Maxwell-Bloch problem in terms of a new set of atomic and field variables that are not very sensitive to limited departures from the ideal limit. For this reason it is not necessary to operate with unrealistically low values of the gain or the mirror transmittivity; (ii) the mean field limit is a good indicator of instabilities and functions as a rough diagnostic tool. This is fortunate because the numerical solutions of the time-dependent Maxwell-Bloch equations require considerable efforts and some guidance can produce significant saving of time. The resonant case, is not very complicated and can be studied exactly with limited effort. For this reason, in this section we limit ourselves to the exact analysis of the resonant laser problem, without any restrictions on the gain of the active medium or the reflectivity of the mirrors. Our starting point is the full set of Maxwell-Bloch equations (2–4) with $\delta_{\rm ac} = 0$. Because the phase of the stationary field is undetermined, it is possible to select $F_{\rm st}(z)$ as a real quantity. In principle, a random fluctuation of the cavity field could force the growth of the imaginary part through a process called phase instability. In this section we simply assume that no phase instability can develop, so that both the field and polarization variables remain real during the linearized evolution. The steady state of this system of equations is given in equations (9, 10, 11). To study the stability of this steady state, we set

$$F(z,t) = F_{\rm st}(z) + {\rm e}^{\lambda t}\delta f(z),$$
$$P(z,t) = P_{\rm st}(z) + {\rm e}^{\lambda t}\delta p(z),$$
$$D(z,t) = D_{\rm st}(z) + {\rm e}^{\lambda t}\delta d(z), \quad (14)$$

into equations (2–4), upon neglecting fluctuation terms of order higher than one. The linearized equation of the field fluctuation takes the form

$$\frac{\rm d}{{\rm d}z}\delta f(z) = M(z)\delta f(z), \quad (15)$$

where

$$M(z) = -\frac{\lambda}{c} + \frac{\lambda+3\gamma_1}{\lambda+\gamma_1}\frac{\alpha F_{\rm st}^2}{1+F_{\rm st}^4} - \frac{\alpha F_{\rm st}^6}{1+F_{\rm st}^4}\frac{2\gamma_1+\lambda}{\lambda+\gamma_1} \times \left(\frac{2\gamma_1\gamma_2}{(\lambda+\gamma_1)(\lambda+\gamma_2)+\gamma_1\gamma_2 F_{\rm st}^4}\right)\cdot \quad (16)$$

The formal solution of equation (15) is

$$\delta f(z) = \delta f(0){\rm e}^{\int_0^z {\rm d}z' M(z')} = \delta f(0){\rm e}^{\Psi(z)}. \quad (17)$$

The problem is that $F_{\rm st}$ is not known in closed analytic form. We can get around this difficulty with a change of independent variable from z to $F_{\rm st}$, if we take advantage of the fact that ${\rm d}z = {\rm d}F_{\rm st}/({\rm d}F_{\rm st}/{\rm d}z)$ and that ${\rm d}F_{\rm st}/{\rm d}z$ is known explicitly from equation (11) and the field fluctuation takes the form

$$\delta f(z,t) = {\rm e}^{\lambda t}\delta f_(z) = {\rm e}^{\lambda t}\delta f(0){\rm e}^{\Psi(z)}. \quad (18)$$

Next, imposing the boundary condition

$$\delta f(0,t) = R\delta f(L, t - \frac{\Lambda - L}{c}), \quad (19)$$

we obtain the characteristic equation

$$\lambda_n = -{\rm i}\alpha_n - \frac{c}{2\Lambda}\frac{(\lambda_n+3\gamma_1)}{\lambda_n+\gamma_1}|\ln R| - \frac{c}{4\Lambda}\frac{\lambda_n+2\gamma_1}{\lambda_n+\gamma_1} \times \left[\ln\left(\frac{(\lambda_n+\gamma_1)(\lambda_n+\gamma_2)+\gamma_1\gamma_2 F_{\rm st}^4(L)}{(\lambda_n+\gamma_1)(\lambda_n+\gamma_2)+\gamma_1\gamma_2 R^2 F_{\rm st}^4(L)}\right)\right], \quad (20)$$

where $\alpha_n = 2\pi nc/\Lambda$. The characteristic equation (20) depends on the cavity linewidth K ($c/\Lambda\gamma_1 = K/|\ln R|$) of the population difference, and the gain of the active medium through the output field intensity $F_{\rm st}^2$. The origin of the term $-{\rm i}\alpha_n$ here can be traced back to the equality $\exp(0) = \exp(2\pi n{\rm i})$ for $n = 0, \pm1, \pm2...$ Note that setting $\exp(0) = 1$ would be a mistake because it would eliminate practically the entire spectrum of eigenvalues. At this point, we have reduced the linearized problem (15) to the solution of an infinite number of characteristic equations, one for each value of α_n.

As instructive example we wish to discuss some limiting cases of the non-linear equation (20) governing the stability of the system.

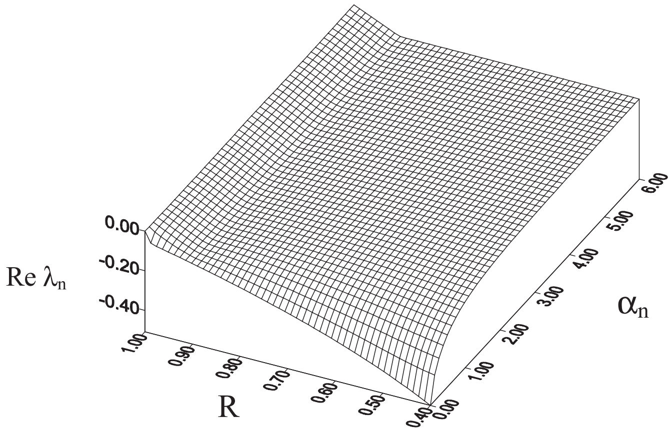

Fig. 1. The largest real parts of the linearized eigenvalues are plotted as functions of α_n and R. We have selected $\alpha L = 1$, $k = 3.55$ and for $\gamma_1 = 10^5$.

(a) Perfect reflectivity is assumed *i.e.* $R = 1$ then the only solution of equation (20) is $\lambda_n = -{\rm i}\alpha_n$ *i.e.* running waves.

(b) When the decay rate of the two-photon polarization is set to be zero, the solution of equation (20) is $\lambda_n = -{\rm i}\alpha_n - \frac{c}{2\Lambda}|\ln R|$ which means a stable mode.

(c) When we set the population relaxation rate γ_2 to be zero, a second order equation in λ_n results. The solution to this equation gives the following acceptable formula for the real part of λ_n

$$\mathrm{Re}\,\lambda_n = -\frac{\gamma_1}{2} - \frac{c}{4\Lambda}|\ln R| + \sqrt{\frac{A+\sqrt{A^2+B^2}}{2}},$$
$$A = \frac{(\gamma_1 + \frac{c}{2\Lambda}|\ln R|)^2}{4} - \frac{\alpha_n^2}{4} - 3\gamma_1\frac{c}{2\Lambda}|\ln R|,$$
$$B = \frac{\alpha_n(\gamma_1 + \frac{c}{2\Lambda}|\ln R|)}{2} - \alpha_n\gamma_1. \quad (21)$$

Investigation of this equation shows that no instabilities develop in this limiting case. This can be understood simply because writing $\gamma_2 = 0$ means a constant population difference and hence no exchange of population between the two levels (see Fig. 1). In this figure we plot the largest real parts of the linearized eigenvalues as functions of α_n and R.

(d) Iterative state, we can solve equation (20) by substituting the value $-{\rm i}\alpha_n$ for λ_n in the right hand side of equation (20). It can be asserted that the highly excited modes α_n such that $n \gg 10$ are stable. This can be seen from looking at the argument of the logarithm in equation (20) in this case with the assumption $\alpha_n > \gamma_2$, $\gamma_1\gamma_2 F_{\rm st}^4$. However when $\gamma_1\gamma_2 F_{\rm st}^4 > (|-{\rm i}\alpha_n + \gamma_1|)(|-{\rm i}\alpha_n + \gamma_2|)$ the mode becomes unstable (see Fig. 2). For increasing α_n the system becomes more stable as the positive ${\rm Re}\lambda_n$-range decreases.

The existence of an infinite number of eigenvalues is not surprising in view of the space-time dependent nature

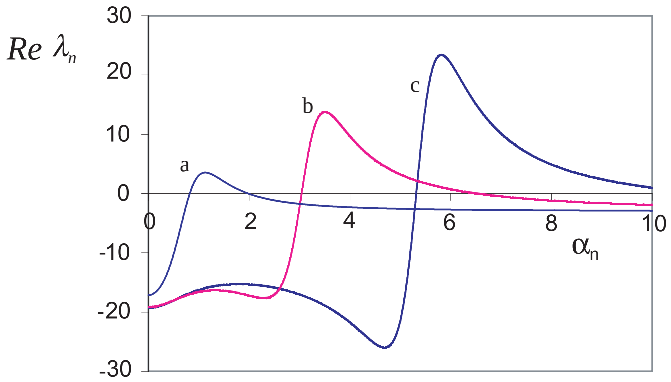

Fig. 2. The largest real parts of the linearized eigenvalues are plotted as functions of α_n viewed as a continuous variable. For all the curves displayed in the figure we have selected $R = 0.8, \bar{\gamma} = 0.1$, $k = 3.55$, and $\alpha L = 1$ (curve a), $\alpha L = 3$ (curve b) and $\alpha L = 5$ (curve c).

of the field and atomic variables and of the boundary conditions of the laser resonator. One is remanded of the ordinary vibration problems, linear string, two-dimensional membrane, etc., except that here we are dealing simultaneously with three fluctuation variables equation (14), and thus on physical grounds, one expects three characteristic roots $\lambda_n^{(1)}$, $\lambda_n^{(2)}$, $\lambda_n^{(3)}$ for each value of n. Because α_n represents the frequency separation between the nth empty cavity resonance and the selected reference mode, it is easy to interpret the set of roots $\lambda_n^{(i)}$, $i = 1, 2, 3$, as descriptive of the growth or decay of an initial fluctuation that develops in correspondence to the nth mode of the cavity. This interpretation forms the basis for a classification of the possible unstable behaviors of the system. If $\text{Re}\lambda_0$ is positive and $\text{Re}\lambda_n$ $(n \neq 0)$ are all negative, an initial fluctuation of the resonant mode will grow exponentially and evolve with the same carrier frequency as the stationary state. Thus, the linearized dynamics of the laser can be described only in terms of the behavior of the resonant mode fluctuation (all the other fluctuations are damped because $\text{Re}\lambda_n < 0$, $n \neq 0$) and the instability will be of the single-mode type. If, on the other hand, $\text{Re}\lambda_0 < 0$ and, for some value of n, $\text{Re}\lambda_n > 0$, the nth cavity mode will support the growth of a fluctuation whose carrier frequency is different from that of the stationary state. Here, the existence of a one-to-one correspondence between the index n, that appears in equation (20), and the longitudinal cavity modes is suggested. This informal suggestion is founded on physical grounds. The main conceptual difficulty with this interpretation is that the notion of "mode" is not well defined when the resonator mirrors have a finite reflectivity, and the elementary cavity excitations have a finite lifetime. In fact, in solving the linearized problem, we have not even introduced resonator eigenfunctions, as one normally would in a standard boundary value problem. For this reason, we continue to refer to $\lambda_n^{(i)}$ as the set of linearized eigenvalues of the nth cavity resonator.

A complete analysis of equation (20), particularly with regard to the role played by the basic laser parameters, gain, intermode spacing, reflectivity and the atomic decay rates, has not been carried out. Equation (20) predicts that both single and multimode unstable behavior can be established with confidence. We begin our analysis by scaling all the relevant rates of the problem to the linewidth γ_1 of the active medium. In this way, equation (20) takes the form

$$\bar{\lambda}_n = -\mathrm{i}\bar{\alpha}_n - \frac{c}{2\gamma_1 \Lambda}\frac{(\bar{\lambda}_n + 3\bar{\gamma})|\ln R|}{\bar{\lambda}_n + 1} - \frac{c}{4\gamma_1 \Lambda}\frac{\bar{\lambda}_n + 2}{\bar{\lambda}_n + 1} \times \left[\ln\left(\frac{(\bar{\lambda}_n + 1)(\bar{\lambda}_n + \bar{\gamma}) + \bar{\gamma}F_{\mathrm{st}}^4(L)}{(\bar{\lambda}_n + 1)(\bar{\lambda}_n + \bar{\gamma}) + \bar{\gamma}R^2F_{\mathrm{st}}^4(L)}\right)\right] \quad (22)$$

where $\bar{\lambda}_n = \lambda_n/\gamma_1$, $\bar{\alpha}_n = \alpha_n/\gamma_1$, and $\bar{\gamma} = \gamma_2/\gamma_1$. The equation (22) can be evaluated by using the iteration method as above (in (d)) by substituting the value $-\mathrm{i}\alpha_n$ for λ_n in the right hand side of equation (22). It can be asserted that the highly excited modes α_n namely $n \gg 10$, are stable. A numerical study of this problem shows that single-mode instabilities $\alpha_n = 0$ tend to be favored in the presence of high gain and laser cavity losses $K > 1$. These conditions are difficult to realize in a practical system. In general, it appears from equation (22) that single-mode instabilities require a scaled cavity linewidth which is sufficiently larger than unity. In order to keep the calculations presented in this paper as realistic as possible, we have chosen to apply our model for a real atomic system, (for the transition $4p_{3/2}$–$6p_{3/2}$ in potassium). The reason for choosing this transition is the result of a compromise. On one hand, one wants the energy of the photons involved to be as large as possible, and preferably in the optical regime. On the other hand, it is hard to find a two-photon transition in the optical regime with a large coupling, since a large two-photon coupling demands the existence of an almost resonant intermediate level with opposite parity. The transition mentioned above involves photons with an energy of $\simeq 7\,980\ \text{cm}^{-1}$ *i.e.* near-infrared, and has a two-photon coupling that is orders of magnitude larger than the other candidates we looked at, due to the almost resonant $5s$ state. Besides the atom, we should also choose a cavity. In the model presented in this paper, we are assuming that only one mode of the cavity field is excited. For this to be true, the cavity should be rather small, since it then supports fewer modes, and these will be better separated in energy. Another advantage of having a small cavity is that the two-photon coupling $\mu^{(2)}$ will be larger, since it is proportional to V^{-1} (following the notation of Loudon) [22], V being the cavity volume. We have chosen the cavity volume $V = 10^{-15}\ \text{m}^3$. Further F and α for the two-photon case will be proportional to $V^{-1/2}$ and $V^{-2/3}$ instead of $V^{-1/4}$ and V^{-1} for the one-photon case. The only one that is decreasing with $(V \ll 1)$ very small cavities is the gain factor.

It is interesting to compare our results with those of previous work where the mean field limit is appeared. When we neglect the spatial dependence, we arrive to equation (4) of reference [19] but taking into account a

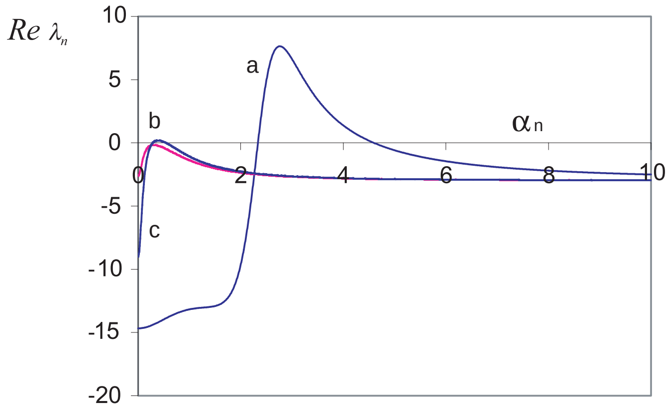

Fig. 3. The largest real parts of the linearized eigenvalues are plotted as functions of α_n viewed as a continuous variable. For all the curves displayed in the figure we have selected $\alpha L = 5$, $k = 0.07$, $\gamma = 0.1$, and $R = 0.3$ (curve a), $R = 0.6$ (curve b) and $R = 0.95$ (curve c).

normalization of the variables $D \to D/D_0$, $\alpha \to kD_0$ and $P \to q/D_0$. Our formulation is based on the conventional Maxwell-Bloch equations, but is distinguished from other treatments by the inclusion of propagation effects, a finite mirror reflectivity and an arbitrary value of the gain parameter. These features make the model more general than the previous studies. An example of the behavior of the eigenvalues in the case of a single-valued state equation is shown in Figure 2. In this figure we plot the largest real parts of the linearized eigenvalues as functions of α_n viewed as a continuous variable. For all the curves displayed in this figure we have selected $R = 0.8$, $\alpha_1 = 100$, $\gamma = 0.1$ and for different values of αL. We show that unstable situation for several values of the relevant parameters (the only physical meaningful values of $\bar{\alpha}_n$ are all the positive and negative multiples of the intermode spacing α_1). Multi-mode instabilities are not bounded by the high-loss requirement, but they still require large values of the gain to reach their threshold. For increasing α_n the system becomes more stable as the positive Reλ_n-range decreases, and the system is completely stable for $\alpha_n > 10$. In Figure 3 the largest real parts of the linearized eigenvalues are plotted as functions of α_n viewed as a continuous variable. With the same parameters as in Figure 2 but for different values of R. This figure gives an example of some typical real parts of the linearized eigenvalues for parameter values that lead to multimode instability. As shown in this figure, the beat frequency due to the superposition of the stationary solution and of the unstable sidebands is sensitive to the value of R. The important feature is the monotonic shift of the positive real parts of the eigenvalues towards higher and higher values of $\bar{\alpha}_n$ for increasing values of the gain. It is important to stress that the existence of unstable off-resonant sidebands requires a good quality cavity in the sense that K must be sufficiently smaller than unity. This is a consequence of the fact that typical instability ranges extend to maximum values of $\bar{\alpha}_n$ of the order of a few units. Thus, if we require that the sideband at $\bar{\alpha}_n = 2\pi nc/L\gamma_1$ be unstable, it is necessary that the ratio $c/L\gamma_1$ be smaller than unity. This can be arranged by selecting large enough values of L or γ_1.

5 Conclusion

We have derived the general Maxwell-Bloch equation for the system consisting of the two-level atoms with dipole forbidden transition, placed in a two-photon one. The treatment has been carried out in the framework of the semiclassical laser theory. We have calculated the spatial behavior of the field strength and have shown the effect of the additional non-linearity due to the two-photon coupling. We have computed the stability analysis of the steady-state solution of the complete Maxwell-Bloch. Although the model is rather idealized, its general features should be relevant to a real single-mode system. The analysis presented in this paper has been inspired by the comparison between the linear stability analysis technique and the so-called weak sideband approach [16]. In our case the linear stability analysis not only agrees with the results of the weak sideband approach, but extends its range of applicability, particularly in the case when the cavity detuning must be taken into account.

The problem we have formulated and solved in this paper has an interesting counterpart in the microwave regime where one can tailor at will, in combination with the choice of the principal quantum number of the pumped Rydberg state. The experimental realization of such a scenario should be relatively easy with present day technology. In our treatment we have focused on the degenerate two-photon laser. It would thus be interesting to study the non-degenerate two-photon laser. We could imagine having a transition in which one photon is visible, and the other is, say, infrared. The frequencies of these two photons could be chosen in such a way that we would obtain a large two-photon coupling and hence this laser type would be easier to realize. In this laser type would also expect Stark shift to play a dominant role. We hope to report on such issues in a forthcoming paper.

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